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THE DENSENESS OF COMPLETE RIEMANNIAN METRICS

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The purpose of this note is to expand a bit on a theorem of K. Nomizu and H. Ozeki. In [1], they proved that on any paracompact connected C^∞ manifold there is a complete Riemannian metric. In fact, it was shown that if M is a C^∞ manifold, T_pM is the tangent space of M at $p \in M$, $\xi, \eta \in T_pM$, and $g_p(\xi, \eta) = g_p = g$ is a Riemannian metric on M, then there is a C^∞ function f on M such that $fg = (fg)_p(\xi, \eta) = f(p)g_p(\xi, \eta)$ is a complete Riemannian metric on M. We intend to prove the following theorem.

Theorem. *Let M be a connected C^∞ manifold with Riemannian metric g. Then given a compact subset $K \subseteq M$, there is a complete Riemannian metric h on M such that $h|_K = g|_K$, where $h|_K$ denotes "h restricted to K".*

Corollary. *If the space of Riemannian metrics on a connected C^∞ manifold M is given the topology of convergence of all derivatives of order up to l on compact subsets of M, then the complete metrics form a dense subset of the space of all metrics. (This is true for each fixed l, $1 \leq l \leq \infty$.)*

We make the following remarks.

Remark 1. The result can clearly be extended to non-connected paracompact manifolds.

Remark 2. In case M is compact, the result is trivial.

Before proceeding with the proof of the theorem, for the convenience of the reader we review the proof of Nomizu-Ozeki. Assume g is not complete, and let $B_p(r)$ be the metric ball

$$B_p(r) = \{q \in M \mid \mu_g(p, q) \leq r\} ,$$

where μ_g is the metric on M arising from the Riemannian metric g. Further let

$$d(p) = \{\sup r \mid B_p(r) \text{ is compact}\} .$$

Then $d: M \to \boldsymbol{R}$ is a continuous real-valued function. It is easy to see that $d(p) > 0$ for all $p \in M$, and it is not difficult to show that there is a C^∞ function $\bar{f}: M \to \boldsymbol{R}^+$ such that $\bar{f}(p) < 1/d(p)$ for all $p \in M$. Let $f = (\bar{f})^2$. Then fg is the required complete Riemannian metric. The proof of the completeness of fg is not difficult and can be found in [1].

We now give the proof of our theorem. Let $M = \cup K_j$, where the K_j are compact, $K_j \subseteq \operatorname{int}(K_{j+1})$, ($\operatorname{int}(N)$ denotes the interior of a subset N of M). If

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K is a given compact subset of M, then $K \subseteq \operatorname{int} K_i$ for some i. For fixed i, the pair $\{\operatorname{int} K_{i+1}, M - K_i\}$ is an open covering of M. By choosing a partition of unity subordinate to this covering, we obtain C^∞ functions ρ_1, ρ_2 on M such that

(a) $\rho_1 \geq 0$, $\rho_2 \geq 0$,

(b) $\rho_1 + \rho_2 \equiv 1$,

(c) $\rho_1 \equiv 1$ on K_i, and $\rho_2 \equiv 1$ on $M - K_{i+1}$,

(d) support $\rho_1 \subseteq \operatorname{int} K_{i+1}$, support $\rho_2 \subseteq M - K_i$.

For the next step we use the theorem of Nomizu and Ozeki to get the existence of a positive C^∞ function f on $M - K_i$ such that the Riemannian metric $f \cdot (g|_{M-K_i}) = h_1$ defines a complete metric μ_{h_1} on each of the connected components of $M - K_i$. Then $h = (\rho_1 + \rho_2 f)g$ is a Riemannian metric on M. We claim μ_h, the induced metric, is complete. If we prove this, we are finished, since $h \equiv g$ on K_i and $K \subseteq K_i$. Let $\{p_\nu\}$ be a μ_h-Cauchy sequence in M. There are two cases. If infinitely many p_ν are in K_{i+1}, then $p_\nu \to p \in K_{i+1}$ since K_{i+1} is compact. If not, almost all p_ν are in $M - K_{i+1}$, and since we can neglect finitely many ν we may assume $\{p_\nu\} \subseteq M - K_{i+1}$. By passing to a subsequence if necessary, we may also assume $\{p_\nu\}$ is contained entirely in a single connected component of $M - K_{i+1}$. (This can be seen as follows. First we may assume that only finitely many p_ν are in K_{i+2}. Otherwise we could conclude the proof of the theorem since K_{i+2} is compact. Let 3σ be the μ_g distance of K_{i+1} from $M - \operatorname{int} K_{i+2}$. Then $\sigma > 0$. Choose N so that $p_\nu \in M - K_{i+2}$ and $\mu_g(p_\nu, p_{\nu+k}) \leq \sigma$ for $\nu \geq N$ and all $k \in Z$, $k \geq 0$. This means given $p_\nu, p_{\nu+k}$, $\nu \geq N$, $k \geq 0$, there is a curve in M connecting $p_\nu, p_{\nu+k}$ with g-length less than 2σ. Since P_ν, $p_{\nu+k} \in M - K_{i+2}$, this curve cannot touch K_{i+1}. Thus p_ν, $p_{\nu+k}$ lie in the same component of $M - K_{i+1}$.) For small enough α, the α neighborhood of K_{i+1}, $U_\alpha(K_{i+1}) = \{p \mid \mu_h(K_{i+1}, p) \leq \alpha\}$ is a compact set. If infinitely many p_ν are in this we are done. So assume $\mu_h(p_\nu, K_{i+1}) > \alpha$ for all ν. Let $\varepsilon < \alpha$. Then any curve of h-length less than ε beginning at some p remains in $M - K_{i+1}$. But on $M - K_{i+1}$, $h = fg = h_1$. Thus the h_1-length of any curve of h-length less than ε beginning at p_ν is equal to the h-length. This proves that the sequence $\{p_\nu\}$ is a μ_{h_1}-Cauchy sequence in $M - K_{i+1} \subseteq M - K_i$. By assumption μ_{h_1} is complete so $\{p_\nu\}$ converges.

Remark 3. If one uses the topology of convergence of k derivatives on compact subsets, then the set of complete metrics is *not* an open subset of the set of all metrics on a non-compact manifold. It may be a residual set, but the author has not tried to prove it.

Bibliography

[1] K. Nomizu & H. Ozeki, *The existence of complete Riemannian metrics*, Proc. Amer. Math. Soc. **12** (1961) 889–891.

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